

THE CONJECTURE OF NOWICKI ON WEITZENBÖCK DERIVATIONS OF POLYNOMIAL ALGEBRAS

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ABSTRACT. The Weitzenböck theorem states that if Δ is a linear locally nilpotent derivation of the polynomial algebra $K[Z] = K[z_1, \dots, z_m]$ over a field K of characteristic 0, then the algebra of constants of Δ is finitely generated. If $m = 2n$ and the Jordan normal form of Δ consists of 2×2 Jordan cells only, we may assume that $K[Z] = K[X, Y]$ and $\Delta(y_i) = x_i$, $\Delta(x_i) = 0$, $i = 1, \dots, n$. Nowicki conjectured that the algebra of constants $K[X, Y]^\Delta$ is generated by x_1, \dots, x_n and $x_i y_j - x_j y_i$, $1 \leq i < j \leq n$. Recently this conjecture was confirmed in the Ph.D. thesis of Khoury, and also by Derksen. In this paper we give an elementary proof of the conjecture of Nowicki. Then we find a very simple system of defining relations of the algebra $K[X, Y]^\Delta$ which corresponds to the reduced Gröbner basis of the related ideal with respect to a suitable admissible order, and present an explicit basis of $K[X, Y]^\Delta$ as a vector space.

INTRODUCTION

Let K be a field of characteristic 0 and let $K[Z] = K[z_1, \dots, z_m]$ be the polynomial K -algebra in m variables. A linear operator Δ of $K[Z]$ is called a derivation if $\Delta(uv) = \Delta(u)v + u\Delta(v)$ for all $u, v \in K[Z]$. The derivation Δ is locally nilpotent if for each $u \in K[Z]$ there exists a $d \geq 1$ such that $\Delta^d(u) = 0$. Locally nilpotent derivations of polynomial algebras are subjects of active investigation. They play essential role in the study of automorphism group of $K[Z]$, including the generation of $\text{Aut } K[x, y]$ by tame automorphisms, the Jacobian conjecture, in invariant theory, Fourteenth Hilbert's problem and other important topics. See the books by Nowicki [N], van den Essen [E], and Freudenburg [F] for details.

The well known theorem of Weitzenböck [W] states that if Δ is a nilpotent linear operator acting on the m -dimensional vector space $KZ = Kz_1 \oplus \dots \oplus Kz_m$ and we extend it to a derivation of $K[Z]$, then the algebra $K[Z]^\Delta$ of constants of Δ , i.e., the kernel of Δ in $K[Z]$, is a finitely generated algebra. We call Δ , which is a locally nilpotent derivation, a Weitzenböck derivation. A modern proof of the theorem of Weitzenböck is given by Seshadri [S], with further simplification by Tyc [T], see also [N, F].

Up to a change of the basis of the vector space KZ , Weitzenböck derivations are determined by their Jordan normal form. Each Jordan cell is an upper triangular matrix with zero diagonal. Hence, for each dimension m it is sufficient to consider a finite number of Weitzenböck derivations Δ . There are algorithms which find a set of generators of $K[Z]^\Delta$ for a given Δ . Nevertheless from computational point

2000 *Mathematics Subject Classification.* 13N15; 13A50; 13P10; 14E07.

The research of the first author was partially supported by Grant MI-1503/2005 of the Bulgarian National Science Fund.

The work of the second author was partially supported by an NSA grant.

of view it is difficult to find explicitly such a system. In particular, no upper bound for the degree of the generators is known.

The algebra of constants of Δ coincides with the algebra of invariants of the linear operator

$$\exp(\Delta) = 1 + \frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \dots$$

and $K[Z]^\Delta$ may be studied also with methods of invariant theory. In particular, when we find the generators of the algebra of invariants of $\exp(\Delta)$, this is stated as the First fundamental theorem of the invariants of $\exp(\Delta)$. When we find the defining relations, this is the Second fundamental theorem.

When the Jordan normal form of the Weitzenböck derivation Δ consists of 2×2 Jordan cells only, we may assume that

$$K[Z] = K[X, Y] = K[x_1, \dots, x_n, y_1, \dots, y_n],$$

$$\Delta = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}$$

and hence the action of Δ is defined by

$$\Delta(x_i) = 0, \quad \Delta(y_i) = x_i, \quad i = 1, \dots, n.$$

Till the end of the paper we shall consider such derivations only. Nowicki [N] conjectured that $K[X, Y]^\Delta$ is generated by x_1, \dots, x_n and the determinants

$$u_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad 1 \leq i < j \leq n.$$

The conjecture of Nowicki was confirmed by Khoury in his Ph.D. thesis [K]. His proof is very computational and uses essentially Gröbner basis techniques. Another proof was given (but not published yet) by Derksen who combined ideas of the proof of Seshadri [S] of the Weitzenböck theorem with the explicit form of the invariants of the special linear group $SL_2(K)$ acting on a direct sum of two-dimensional $SL_2(K)$ -invariant vector spaces.

The purpose of our paper is to give a new elementary proof of the conjecture of Nowicki. We use easy arguments from undergraduate algebra and a simple induction only. We find also a uniformly looking explicit set of defining relations of the algebra of constants $K[X, Y]^\Delta$ which corresponds to the reduced Gröbner basis of the related ideal of $K[X, U]$, where $U = \{u_{ij} \mid 1 \leq i < j \leq n\}$, as well as a basis of $K[X, Y]^\Delta$ as a vector space.

1. THE CONJECTURE OF NOWICKI

We fix the sets of variables

$$X' = \{x_1, \dots, x_{n-1}\}, \quad Y' = \{y_1, \dots, y_{n-1}\}$$

$$X = X' \cup \{x_n\}, \quad Y = Y' \cup \{y_n\}$$

and the derivation of $K[X, Y] = K[X', Y', x_n, y_n]$

$$\Delta = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}.$$

The derivation Δ acts in the same manner on all x_i and y_i . Hence the K -algebra endomorphism φ_α of $K[X, Y]$, $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in K^{n-1}$, defined by

$$\varphi_\alpha(x_i) = x_i, \quad \varphi_\alpha(y_i) = y_i, \quad i = 1, \dots, n-1,$$

$$\varphi_\alpha(x_n) = \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}, \quad \varphi_\alpha(y_n) = \alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1},$$

commutes with Δ . If $f(X', Y', x_n, y_n)$ belongs to the algebra of constants $K[X, Y]^\Delta$, then

$$\varphi_\alpha(f) = f(\varphi_\alpha(X'), \varphi_\alpha(Y'), \varphi_\alpha(x_n), \varphi_\alpha(y_n))$$

$$= f(X', Y', \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}, \alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) \in K[X', Y']^\Delta$$

for all $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in K^{n-1}$.

Lemma 1. *Let $n \geq 2$ and let a nonzero polynomial $f = f(X', Y', x_n, y_n)$ be homogeneous with respect to x_n, y_n . If*

$$\varphi_\alpha(f) = f(X', Y', \alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}, \alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) = 0$$

for an $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in K^{n-1} \setminus 0$, then f is divisible by

$$w_\alpha(X', Y', x_n, y_n) = (\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) y_n - (\alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) x_n.$$

Proof. Let

$$0 \neq f = a_p y_n^p + a_{p-1} x_n y_n^{p-1} + \dots + a_1 x_n^{p-1} y_n + a_0 x_n^p, \quad a_i \in K[X', Y'],$$

be a polynomial homogeneous in x_n, y_n . Then

$$(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) f = a_p w_\alpha y_n^{p-1}$$

$$+ a_p (\alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) x_n y_n^{p-1} + (\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) x_n g,$$

where $g = a_{p-1} y_n^{p-1} + a_{p-2} x_n y_n^{p-2} + \dots + a_1 x_n^{p-2} y_n + a_0 x_n^{p-1}$. Since $\varphi_\alpha(a_p(\alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) y_n^{p-1} + g) = 0$ we can assume by induction on the degree of a polynomial relative to x_n, y_n that w_α divides $a_p(\alpha_1 y_1 + \dots + \alpha_{n-1} y_{n-1}) y_n^{p-1} + g$ (the base of induction for the degree equal to zero is obviously correct) and so w_α divides $(\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}) f$.

The polynomial $w_\alpha(X', Y', x_n, y_n)$ is irreducible in $K[X', Y', x_n, y_n]$. Since $\alpha_1 x_1 + \dots + \alpha_{n-1} x_{n-1}$ does not depend on x_n, y_n , we conclude that $w_\alpha(X', Y', x_n, y_n)$ divides $f(X', Y', x_n, y_n)$ in $K[X', Y', x_n, y_n]$. \square

Corollary 2. *If $\varphi_\alpha(f) = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in K^{n-1} \setminus 0$, then $f = 0$ if $n > 2$ and f is divisible by $u_{12} = x_1 y_2 - x_2 y_1$ if $n = 2$.*

Explanation. If $n > 2$ then f is divisible by $(x_1 + \alpha_2 x_2) y_n - (y_1 + \alpha_2 y_2) x_n$ where α_2 is any element of K . So f is divisible by infinitely many pairwise non-proportional irreducible polynomials. This is of course impossible if $f \neq 0$. If $n = 2$ then by Lemma 1, f is divisible by u_{12} and in this case all w_α with non-zero α are proportional to u_{12} . \square

Theorem 3. *Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, $n \geq 1$, and let Δ be the Weitzenböck derivation of $K[X, Y]$ defined by*

$$\Delta = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}.$$

The algebra of constants $K[X, Y]^\Delta$ is generated by

$$x_i, \quad i = 1, \dots, n, \quad u_{ij} = x_i y_j - x_j y_i, \quad 1 \leq i < j \leq n.$$

Proof. Let

$$f = f(X', Y', x_n, y_n) \in K[X, Y]^\Delta.$$

For a monomial $v \in K[X, Y]$ which is of degree (d_1, d_2) with respect to (X, Y) , i.e., of degree d_1 in X and d_2 in Y , the image $\Delta(v)$ is of degree $(d_1 + 1, d_2 - 1)$. Hence, if $f \in K[X, Y]^\Delta$, then the homogeneous in (X, Y) components of f are also in $K[X, Y]^\Delta$ and we may assume that f is homogeneous in (X, Y) . Similarly, if a monomial u is of total degree p in x_n, y_n , the same is true for $\Delta(u)$. Again, we may assume that f is homogeneous in x_n, y_n . We shall prove the theorem by induction on n and on the total degree in x_n, y_n . If $n = 1$, then $K[x_1, y_1]^\Delta = K[x_1]$. We assume that $n > 1$ and $K[X', Y']^\Delta$ is generated by X' and $U' = \{u_{ij} \mid 1 \leq i < j \leq n-1\}$. Since $\deg_X(u_{ij}) = \deg_Y(u_{ij}) = 1$ and $1 = \deg_X(x_i) > \deg_Y(x_i) = 0$, the homogeneous in (X, Y) elements $v \in K[X', Y']^\Delta$ satisfy $\deg_{X'}(v) \geq \deg_{Y'}(v)$. If $\deg_{x_n, y_n}(f) = 0$, then $f = f(X', Y') \in K[X', Y']^\Delta$. Hence we may consider the case $\deg_{x_n, y_n}(f) > 0$. We write f in the form

$$f = (a_p y_n^p + a_{p-1} x_n y_n^{p-1} + \cdots + a_1 x_n^{p-1} y_n + a_0 x_n^p) x_n^q,$$

where $a_j = a_j(X', Y') \in K[X', Y']$ and $a_p \neq 0$. Since both f and x_n^q belong to $K[X, Y]^\Delta$, the same holds for f/x_n^q and we may assume that $q = 0$. The next observation is that $a_p \in K[X', Y']^\Delta$ because

$$0 = \Delta(f) = \Delta(a_p) y_n^p + (p a_p + \Delta(a_{p-1})) x_n y_n^{p-1} + \cdots + (a_1 + \Delta(a_0)) x_n^p.$$

Hence a_p has the form

$$a_p = \sum x_{s_1} \cdots x_{s_k} b_s(U').$$

If $\varphi_\alpha(f) = 0$ for all $\alpha \in K^{n-1}$, then Corollary 2 gives that $n = 2$ and f is divisible by u_{12} . Hence $f_1 = f/u_{12}$ also belongs to $K[X, Y]^\Delta$. Since $\deg_{x_2, y_2}(f_1) = \deg_{x_2, y_2}(f) - 1$, we apply inductive arguments and conclude that $f \in K[x_1, x_2, u_{12}]$. Now we consider the case when $\varphi_\alpha(f) \neq 0$ for some $\alpha \in K^{n-1}$. The operators φ_α and Δ commute, so we have that $\varphi_\alpha(f) \in K[X', Y']^\Delta$. Also clearly

$$\deg_X(f) = \deg_{X'}(\varphi_\alpha(f)) \geq \deg_{Y'}(\varphi_\alpha(f)) = \deg_Y(f).$$

Hence

$$\deg_X(f) = \deg_{X'}(a_p) \geq \deg_Y(f) = p + \deg_{Y'}(a_p) \text{ and}$$

$$f = \sum x_{s_1} \cdots x_{s_p} c_s(X', U') y_n^p + \sum_{i=0}^{p-1} a_i(X', Y') x_n^{p-i} y_n^i.$$

We rewrite $x_{s_j} y_n$ in the form

$$x_{s_j} y_n = (x_{s_j} y_n - x_n y_{s_j}) + x_n y_{s_j} = u_{s_j n} + x_n y_{s_j}$$

and obtain

$$x_{s_1} \cdots x_{s_p} y_n^p = \prod_{j=1}^p (u_{s_j n} + x_n y_{s_j}) = \prod_{j=1}^p u_{s_j n} + x_n g(X', Y', x_n, y_n)$$

for some $g(X', Y', x_n, y_n) \in K[X', Y', x_n, y_n]$. Hence

$$f = \sum c_s(X', U') \left(\prod_{j=1}^p u_{s_j n} + x_n g(X', Y', x_n, y_n) \right) + \sum_{i=0}^{p-1} a_i(X', Y') x_n^{p-i} y_n^i$$

$$= \sum c_s(X', U') \prod_{j=1}^p u_{s_j n} + x_n f_1(X', Y', x_n, y_n)$$

for some $f_1(X', Y', x_n, y_n) \in K[X', Y', x_n, y_n]$. Since $f, x_n, c_s(X', U') \prod_{j=1}^p u_{s_j n} \in K[X, Y]^\Delta$, the same is true for $f_1(X', Y', x_n, y_n)$ and we apply induction on the degree of f_1 in x_n, y_n . This completes the proof of the theorem. \square

2. DEFINING RELATIONS

In this section we shall show that the defining relations of the algebra of constants $K[X, Y]^\Delta$, with respect to the generators $x_i, u_{ij} = x_i y_j - x_j y_i$ from the conjecture of Nowicki, consists of the relations

$$r(i, j, k, l) = u_{ij} u_{kl} - u_{ik} u_{jl} + u_{il} u_{jk} = 0, \quad 1 \leq i < j < k < l \leq n,$$

$$s(i, j, k) = x_i u_{jk} - x_j u_{ik} + x_k u_{ij} = 0, \quad 1 \leq i < j < k \leq n.$$

Our proof will give that the elements $r(i, j, k, l)$ and $s(i, j, k)$ form the reduced Gröbner basis of the corresponding ideal of $K[X, U]$, $U = \{u_{ij} \mid 1 \leq i < j \leq n\}$, with respect to a suitable admissible order. This provides also a basis of $K[X, Y]^\Delta$ as a vector space.

It will be convenient to identify the generator $u_{ij} \in U$ with the open interval (i, j) on the real line and define the interval length of u_{ij} by $|u_{ij}| = j - i$. We say that u_{ij} and u_{kl} intersect each other if the intervals (i, j) and (k, l) have a nonempty intersection and are not contained in each other. This means that one of the inequalities $i < k < j < l$ and $k < i < l < j$ holds. We say also that u_{ij} covers x_k if k belongs to the open interval (i, j) .

We summarize the necessary background on Gröbner bases. For more details we refer for example to the book by Adams and Loustaunau [AL].

Let $Z = \{z_1, \dots, z_m\}$ be a set of variables. The linear ordering \succ on the set of monomials $[Z]$ is admissible, if it satisfies the descending chain condition and $u \succ v$ for $u, v \in [Z]$ implies $uw \succ vw$ for all $w \in [Z]$. Every nonzero polynomial $f(Z)$ of $K[Z]$ can be written as

$$f = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_k v_k, \quad 0 \neq \beta_j \in K,$$

where $v_1 \succ v_2 \succ \dots \succ v_k$. We denote by \bar{f} the leading monomial v_1 of f in $[Z]$.

Let J be an ideal of $K[Z]$ and let $I(J)$ be the set of leading monomials of J . A generating set G of J is called a Gröbner basis of J (with respect to the fixed admissible order on $[Z]$) if for any $f \in J$ there exists an $f_i \in G$ such that \bar{f} is divisible by \bar{f}_i . Equivalently, the set $I(G)$ of leading monomials of G generates the semigroup ideal $I(J)$ of $[Z]$. A subset G of J is a Gröbner basis of J if and only if it has the following property. The set of normal monomials of $[Z]$ with respect to G , i.e., the monomials which are not divisible by an element of $I(G)$, forms a K -basis of the factor algebra $K[Z]/J$. The set of normal monomials with respect to G spans $K[Z]/J$ for any subset G of J . Hence $G \subset J$ is a Gröbner basis of J if and only if the set of normal monomials with respect to G is linearly independent in the factor algebra $K[Z]/J$. The Gröbner basis G is reduced if the monomials v_i participating in each $f_j \in G$ are not divisible by the leading monomials of $G \setminus \{f_j\}$.

It is easy to see that the following ordering is admissible. We believe that it may be applied also to other problems.

Definition 4. We order the monomials of $K[X, U]$ first by the degree in X and U , then by the total interval length of the participating variables u_{ij} and finally lexicographically, as follows.

Let

$$v = x_{i_1} \cdots x_{i_c} u_{j_1 k_1} \cdots u_{j_d k_d},$$

where $i_1 \leq \cdots \leq i_c$, $j_1 \leq \cdots \leq j_d$ and $k_a \leq k_{a+1}$ if $j_a = j_{a+1}$, and

$$v' = x_{i'_1} \cdots x_{i'_c} u_{j'_1 k'_1} \cdots u_{j'_d k'_d},$$

with similar restrictions on i'_a, j'_b, k'_b .

We define $v \succ v'$ if

- (i) $c > c'$ (the degree of v in X is bigger than the degree of v' in X);
- (ii) $c = c'$ and $d > d'$ (the degree of v in U is bigger than the degree of v' in U);
- (iii) $c = c'$, $d = d'$ and

$$\sum_{b=1}^d |u_{j_b k_b}| > \sum_{b=1}^d |u_{j'_b k'_b}|$$

(the total interval length of v is bigger than that of v');

- (iv) $c = c'$, $d = d'$,

$$\sum_{b=1}^d |u_{j_b k_b}| = \sum_{b=1}^d |u_{j'_b k'_b}|$$

and $\omega \succ \omega'$ for the $(c + 2d)$ -tuples

$$\omega = (i_1, \dots, i_c, j_1, \dots, j_d, k_1, \dots, k_d), \quad \omega' = (i'_1, \dots, i'_c, j'_1, \dots, j'_d, k'_1, \dots, k'_d),$$

where $(a_1, \dots, a_p) \succ (b_1, \dots, b_p)$ if $a_1 = b_1, \dots, a_e = b_e, a_{e+1} < b_{e+1}$ for some e .

We call this ordering degree–interval length–lexicographic order (DILL order) of $K[X, U]$.

Theorem 5. Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, $n \geq 1$, and let Δ be the Weitzenböck derivation of $K[X, Y]$ defined by

$$\Delta = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}.$$

- (i) The algebra of constants has the presentation

$$K[X, Y]^\Delta \cong K[X, U]/(R, S),$$

where $X = \{x_i \mid i = 1, \dots, n\}$, $U = \{u_{ij} \mid 1 \leq i < j \leq n\}$ and the ideal (R, S) is generated by

$$R = \{r(i, j, k, l) = u_{ij}u_{kl} - u_{ik}u_{jl} + u_{il}u_{jk} \mid 1 \leq i < j < k < l \leq n\},$$

$$S = \{s(i, j, k) = x_i u_{jk} - x_j u_{ik} + x_k u_{ij} \mid 1 \leq i < j < k \leq n\}.$$

- (ii) The set $R \cup S$ is the reduced Gröbner basis of the ideal (R, S) with respect to the DILL order of $K[X, U]$.

- (iii) As a vector space $K[X, Y]^\Delta$ has a basis consisting of all products

$$x_{i_1} \cdots x_{i_c} u_{j_1 k_1} \cdots u_{j_d k_d}$$

such that the generators $u_{j_p k_p}$ and $u_{j_q k_q}$ do not intersect each other and $u_{j_p k_p}$ does not cover x_{i_t} for any p, q, t .

Proof. Let $\pi : K[X, U] \rightarrow K[X, Y]^\Delta$ be the homomorphism defined by

$$\pi(x_i) = x_i, \quad \pi(u_{jk}) = x_j y_k - x_k y_j,$$

and let $J = \text{Ker}(\pi) \subset K[X, U]$. We shall prove the following:

- (1) The ideal J contains R and S ;
- (2) The set of normal monomials with respect to $R \cup S$ coincides with the set of all products

$$x_{i_1} \cdots x_{i_c} u_{j_1 k_1} \cdots u_{j_d k_d}$$

such that the generators $u_{j_p k_p}$ and $u_{j_q k_q}$ do not intersect each other and $u_{j_p k_p}$ does not cover x_{i_t} for any p, q, t ;

- (3) The images of these normal monomials under π are linearly independent in $K[X, Y]$.

Since $K[X, Y]$ contains $K[X, Y]^\Delta = \pi(K[X, U]) \cong K[X, U]/J$, the statement (3) implies that the images of normal monomials under π are linearly independent in $K[X, U]/J$ and so, as we mentioned above, $R \cup S$ is a Gröbner basis of $J = \text{Ker}(\pi)$. We also check that this basis is reduced.

Step 1: It is easy to verify directly that $\pi(s(i, j, k)) = \pi(r(i, j, k, l)) = 0$. Also

$$\pi(s(i, j, k)) = \det \begin{pmatrix} x_i & x_j & x_k \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix}$$

expanded relative to the first row and

$$2\pi(r(i, j, k, l)) = \det \begin{pmatrix} x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \\ x_i & x_j & x_k & x_l \\ y_i & y_j & y_k & y_l \end{pmatrix}$$

expanded relative to the first two rows.

Step 2: If $1 \leq i < j < k < l \leq n$ then $\overline{r(i, j, k, l)} = u_{ik} u_{jl}$ since $|u_{ik} u_{jl}| = |u_{il} u_{jk}| > |u_{ij} u_{kl}|$ and $(i, j, k, l) \succ (i, j, l, k)$. Similarly, if $1 \leq i < j < k \leq n$ then $\overline{s(i, j, k)} = x_j u_{ik}$ since $|u_{ik}| > |u_{jk}|$ and $|u_{ik}| > |u_{ij}|$. So different leading monomials do not divide each other and the set $R \cup S$ is reduced.

Let

$$v = x_{i_1} \cdots x_{i_c} u_{j_1 k_1} \cdots u_{j_d k_d}$$

be a normal monomial with respect to $R \cup S$. If two variables u_{ik} and u_{jl} intersect each other, i.e., either $1 \leq i < j < k < l \leq n$ or $1 \leq j < i < l < k \leq n$, then their product $u_{ik} u_{jl}$ is the leading monomial of $r(i, j, k, l)$ or $r(k, l, i, j)$ accordingly. Hence in the normal monomial v the participating generators $u_{j_p k_p}$ and $u_{j_q k_q}$ do not intersect each other. Again, if u_{ik} covers x_j , i.e., $i < j < k$, then $x_j u_{ik}$ is the leading monomial of $s(i, j, k)$. Hence $u_{j_p k_p}$ does not cover x_{i_t} in the normal monomial v for any p, q, t .

Step 3: We want to show that the images of the normal monomials under π are linearly independent in $K[X, Y]$.

Introduce the following ordering on the monomials of $K[X, Y]$:

$$x_1^{a_1} y_1^{b_1} \cdots x_n^{a_n} y_n^{b_n} > x_1^{a'_1} y_1^{b'_1} \cdots x_n^{a'_n} y_n^{b'_n},$$

if $(a_1, b_1, \dots, a_n, b_n) > (a'_1, b'_1, \dots, a'_n, b'_n)$ lexicographically, i.e., either $a_1 > a'_1$ or $a_1 = a'_1, b_1 = b'_1, \dots, b_k = b'_k$, and either $a_{k+1} > a'_{k+1}$ or $a_{k+1} = a'_{k+1}, b_{k+1} > b'_{k+1}$ for some k . The leading monomial of $\pi(u_{jk}) = x_j y_k - x_k y_j$, $j < k$, in $K[X, Y]$

is $\text{lead}(u_{jk}) = x_j y_k$ since $x_j^1 y_j^0 x_k^0 y_k^1 > x_j^0 y_j^1 x_k^1 y_k^0$ and the leading monomial of $w = x_{i_1} \cdots x_{i_c} u_{j_1 k_1} \cdots u_{j_d k_d}$ is

$$\text{lead}(w) = x_{i_1} \cdots x_{i_c} x_{j_1} y_{k_1} \cdots x_{j_d} y_{k_d},$$

so $\deg_Y(\pi(w)) = \deg_U(w)$.

We shall prove by induction on $\deg_Y(\pi(w))$ that a normal monomials $w \in K[X, U]$ is uniquely determined by its leading monomial $\text{lead}(w)$.

If $\deg_Y(\pi(w)) = 0$ then $\deg_U(w) = 0$ and $\text{lead}(w) = w$. If $\deg_Y(\pi(w)) > 0$ let

$$\text{lead}(w) = x_1^{a_1} \cdots x_k^{a_k} y_k^{b_k} \cdots x_n^{a_n} y_n^{b_n}$$

where k is the smallest number for which $b_k \neq 0$. Hence w contains u_{ik} for some $i < k$. Choose the largest number j for which w contains u_{jk} . Then $j < k$ and must be the largest among these numbers with $a_j \neq 0$.

Indeed, w is a normal monomial. Assume that exists an l for which $a_l \neq 0$ and $j < l < k$. Then w cannot contain x_l since x_l is covered by u_{jk} . But it also cannot contain u_{lm} since $m \geq k$ by the choice of k and if $m = k$ then w contains u_{lk} contrary to the choice of j ; so $m > k$ and u_{jk} and u_{lm} would intersect each other.

Therefore $w = u_{jk} w_1$ where u_{jk} is uniquely determined by w . By induction $\text{lead}(w_1)$ determines w_1 and so w is determined by $\text{lead}(w)$. This, of course, implies that $\text{lead}(w)$ are different for different normal monomials and hence the images $\pi(w)$ of these monomials are linearly independent. □

ACKNOWLEDGEMENTS

The first author is grateful to the Department of Mathematics of the Wayne State University in Detroit for the warm hospitality during his visit when most of this work was carried out.

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